Zeta-regularized Determinants of Laplacians on Polygons

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(joint work with Clara Aldana and Werner Mueller)

The main result presented here is Theorem 6 which gives an explicit formula for the variation of the derivative of the spectral zeta function at zero for any convex polygonal domain. In forthcoming work [1], we shall use this to derive an explicit formula for the zeta-regularized determinant of the Laplacian. Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain with n sides. The Euclidean Laplacian Δ_Ω on Ω with Dirichlet boundary condition has eigenvalues

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$$

By Weyl's Law, the spectral zeta function

$$\zeta_\Omega(s) := \sum_{k=1}^{\infty} \lambda_k^{-s}$$

is holomorphic on the half plane $\{\Re s > 1\}$. The heat trace

$$TrH_\Omega(t) = \sum_{k=1}^{\infty} e^{-\lambda_k t},$$

is related to the zeta function by

$$\zeta_\Omega(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} TrH_\Omega(t) dt. \tag{0.1}$$

The heat trace admits an asymptotic expansion as time tends to zero computed in [3]

$$TrH_\Omega(t) \sim \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8\sqrt{\pi t}} + \sum_{i=1}^{n} \frac{\pi^2 - \alpha_i^2}{24\pi\alpha_i} + O(e^{-c/t}). \tag{0.2}$$

Above, α_i is the interior angle at the i^{th} vertex, and $|\Omega|$, $|\partial\Omega|$ denote respectively the area of Ω and the length of the boundary $\partial\Omega$. We note that the constant c is bounded below by a constant computed in [5].

It follows from (0.1), (0.2), and the meromorphic continuation of the Gamma function that ζ admits a meromorphic continuation to the complex plane which is regular at 0. The *zeta-regularized determinant* is defined to be

$$\det(\Delta_\Omega) = e^{-\zeta'_\Omega(0)}.$$

It is straightforward to compute that $\zeta(0)$ is the coefficient of t^0 in (0.2). Consequently, for $\lambda \in (0, \infty)$, the zeta function transforms under scaling of the domain by λ as follows

$$\zeta'_{\lambda\Omega}(0) = \zeta_\Omega(0) \log \lambda + \zeta'_\Omega(0).$$

The determinant therefore scales by

$$\det(\Delta_{c\Omega}) = c^{-\zeta_\Omega(0)} \det(\Delta_\Omega).$$

For *smoothly* bounded domains, the coefficient of t^0 in the small-time asymptotic expansion of the heat trace is a *topological invariant,* namely one sixth of the Euler characteristic [3]. Therefore, the extrema of the determinant are well defined on convex smoothly bounded domains of fixed area. For polygons, this is no longer the case.

Lemma 1. *Let R be a convex n-gon with all angles equal, and let P be a convex n-gon whose angles are not all equal. Assume R and P both have unit area. Then,*

(1) $\exists\, a > 0$ *such that* $\det(\Delta_{aR}) > \det(\Delta_{aP})$,
(2) $\exists\, b > 0$ *such that* $\det(\Delta_{bR}) < \det(\Delta_{bP})$,
(3) $\exists\, c > 0$ *such that* $\det(\Delta_{cR}) = \det(\Delta_{cP})$.

The proof is a straightforward calculation and is left to the reader. □

We are therefore motivated to define a spectral invariant which is *well-defined* on the moduli space of convex n-gons.

Proposition 0.1. *Let $\mathbb{M}_n$ be the moduli space of convex n-gons, which is the space of all similarity classes of convex n-gons. Then, the following function is well defined on $\mathbb{M}_n$.*

$$f(P) = Z_P'(0) - \frac{1}{2} Z_P(0) \log Area(P), \quad P \in \mathbb{M}_n.$$

For more details, see [1].

1. Preliminary variational formulae

Consider a conformal variation of the Euclidean metric $g \mapsto e^{2\sigma} g$, where σ is a smooth function. A computation analogous to those in [4] gives the following variational formula for $\zeta'(0)$,

$$\delta\zeta'(0) = -\gamma\delta\zeta(0) + C(\sigma), \tag{1.1}$$

where γ is Euler's constant, $\gamma = \Gamma'(1)$, and $C(\sigma)$ is the constant term in the trace of $2\delta\sigma H$. The coefficients in the short-time asymptotic expansion of the heat trace (0.2) can be computed by integrating a corresponding *local expansion* defined by the curvature and its derivatives [3]. To compute $C(\sigma)$, we may integrate the product of $2\delta\sigma$ with the *local heat trace expansion.* Since the curvature vanishes identically away from the corners, only the corners contribute to $C(\sigma)$. We may therefore compute the contribution to $C(\sigma)$ from each vertex and sum over the vertices.

A fixed half-strip with the standard Euclidean metric,

$$T = (-\infty, 0]_x \times [0,1]_y, \quad g_{Eucl} = dx^2 + dy^2,$$

can be conformally mapped onto the sector

$$S = (0, \alpha^{-1} e^{\lambda}]_r \times [0, \alpha]_\phi, \quad g_{Eucl} = dr^2 + r^2 d\phi^2 = e^{2\sigma}(dx^2 + dy^2),$$

where the coordinates and conformal factor σ are

$$y = \frac{\phi}{\alpha}, \quad x = \frac{\log(r\alpha) - \lambda}{\alpha}, \quad \sigma = x\alpha + \lambda.$$

The conformal factor σ is a smooth function of x which depends on the two parameters, α and λ. The total differential of the conformal factor σ with respect to the parameters α and λ,

$$\delta\sigma = x\ \mathbf{d}\alpha +\ \mathbf{d}\lambda = \frac{\log r + \log\alpha - \lambda}{\alpha}\ \mathbf{d}\alpha +\ \mathbf{d}\lambda.$$

Formula (2.5) in [5] gives the Green's function for a sector of opening angle α (see also [3] p. 44). The inverse Laplace transform, denoted L^{-1}, of the Green's function is the heat kernel. Let

$$C(\alpha) = \int_0^\infty r\log r dr \int_0^\alpha d\alpha L^{-1}\left\{\frac{1}{\pi^2}\int_0^\infty K_{ix}^2(r\sqrt{s})\frac{\sinh(\pi-\alpha)x}{\sinh\alpha x}dx\right\},$$

and let

$$A(\alpha) := -\frac{\pi\log\alpha}{12\alpha} - \frac{\pi}{12\alpha} - \frac{\alpha\log\alpha}{12\pi} + \frac{\alpha}{12\pi} - \gamma\frac{\pi^2-\alpha^2}{24\pi\alpha}.$$

The angles and side lengths of a convex polygonal domain P cannot be varied independently, but must satisfy certain *constraints.* The interior angles $\{\alpha_i\}_{i=1}^n$ must sum to $\pi(n-2)$, and the scale factors at each vertex $\{\lambda_i\}_{i=1}^n$ together with a global scale factor λ_0 are related by

$$\lambda_i = \lambda_0 - \sum_{j\neq i}(\pi-\alpha_i)\log|p_i - p_j|,$$

where the points $\{p_j\}_{j=1}^n$ lie on the unit circle. We then have the following.

Theorem 6. *Let P be a convex n-gon in the plane with interior angles $\{\alpha_i\}_{i=1}^n$, and let $A(\alpha_i)$, $C(\alpha_i)$, and λ_i be defined as above. For a conformal variation of P which maps P onto a Euclidean n-gon, the conformal variation of $\zeta_P'(0)$ is*

$$\sum_{i=1}^n \frac{C(\alpha_i)}{\alpha_i}\ \boldsymbol{d}\alpha + \delta\left(\sum_{i=1}^n A(\alpha_i) + \left(\frac{\pi\lambda_i}{12\alpha_i}\right) - \frac{1}{12\pi}(\alpha_i\lambda_i) + \left(\frac{1}{12\pi}(1-\alpha_i)(\lambda_i-\lambda_0)\right)\right).$$

In forthcoming work, we use Theorem 1 to compute an explicit formula for the function f defined in Proposition 1 and study the extrema of this spectral invariant in the spirit of [4]. We shall also apply our results to surfaces with conical singularities. Useful ideas for this work were inspired by [2]; the above result is a correction of a similar formula in [2].

References

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